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1997 J. Phys.: Condens. Matter 9 5117

(http://iopscience.iop.org/0953-8984/9/24/011)

J. Phys.: Condens. Matter **9** (1997) 5117–5135. Printed in the UK PII: S0953-8984(97)82153-4

Decoupling of layered superconducting films in parallel magnetic field

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Received 25 February 1997

Abstract. The issue of the decoupling of extreme type-II superconducting thin films ($\lambda_L \to \infty$) with weakly Josephson-coupled layers in magnetic field parallel to the layers is considered via the corresponding frustrated XY model used to describe the mixed phase in the critical regime. For the general case of arbitrary field orientations such that the perpendicular magnetic field component is larger than the decoupling crossover scale characteristic of layered superconductors, we obtain independent parallel and perpendicular vortex lattices. Specializing to the double-layer case, we compute the parallel lower-critical field with entropic effects included, and find that it vanishes exponentially as the temperature approaches the layer decoupling transition in zero field. The parallel reversible magnetization is also calculated in this case, where we find that it shows a crossover phenomenon as a function of parallel field in the intermediate regime of the mixed phase in lieu of a true layer-decoupling transition. It is argued that such is the case for any finite number of layers, since the isolated double layer represents the weakest link.

1. Introduction

The study of layered superconductivity has been reinvigorated by the discovery of the high-temperature oxide superconductors [1]. In the case of the bismuth (Bi) and thallium (Tl) based compounds, for example, the superconducting coherence length ξ_c perpendicular to the conducting planes is less than the separation between layers, and London theory fails [1]. Hence a Lawrence–Doniach (LD) type of description in terms of weakly Josephson-coupled layers becomes necessary [2]. The following question then naturally arises: Does a layer decoupling transition, aided perhaps by the introduction of a parallel magnetic field [3], occur in such systems in addition to or in place of the conventional type-II superconducting ones marked by the lower and upper critical fields [4], $H_{c1}(T)$ and $H_{c2}(T)$?

In the absence of external magnetic field parallel to the layers, a layer decoupling transition is indeed predicted to exist theoretically at a critical temperature T_* that lies above that of the intra-layer Kosterlitz–Thouless (KT) transition temperature, T_c [5–8]. This result is based on studies of the XY model with weakly coupled layers, which accurately describes a layered superconductor in the absence of fluctuations of the magnetic field [9, 10]; for example, in the intermediate regime of the mixed phase found in extreme type-II superconductors, where it is appropriate to take the limit $\lambda_L \to \infty$ for the London penetration length λ_L. Also, recent experiments on Bi-based high-temperature superconductors find evidence for a superconducting transition at T_c^c, where the c-axis

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resistivity vanishes, that lies a few tenths of a degree Kelvin above a second superconducting transition temperature T_c^{ab}, where the resistivity in the ab-planes vanishes [11, 12]. Hence both theory and experiment find an extraordinary regime in temperature, $T_c = T_c^{ab} < T < T_* = T_c^c$, where Josephson tunnelling between layers exists while the layers themselves are *resistive* and show no intra-layer phase coherence. (It has recently been argued in [13] that T_c^c in fact marks a sharp crossover for the bulk case.) Finally, it is worth pointing out that Monte Carlo simulations of the three-dimensional (3D) XY model obtain a similar behaviour in the presence of a large magnetic field perpendicular to the layers [10].

The nature of layer decoupling in the presence of magnetic field parallel to the layers, on the other hand, is less well understood. Early work by Efetov suggests that superconducting layers decouple at high parallel fields $B_\parallel > B_*^\parallel(0) \sim \Phi_0/d^2\gamma$, where Φ_0 denotes the flux quantum, $\gamma = (m_c/m_{ab})^{1/2}$ is the mass anisotropy parameter, and d denotes the separation between layers [3]. The latter calculation is based on a high-temperature series expansion analysis of the LD model. A recent study of the same model by Korshunov and Larkin that employs a Coulomb gas representation, however, finds no such layer decoupling in the high-field limit for temperatures below the decoupling transition [14]. In this scenario, therefore, the layers remain effectively Josephson coupled up to the parallel upper-critical field, $H_{c2}^\parallel(T)$.

In this paper, we shall also examine the decoupling of layered superconductors in parallel magnetic field, but in film geometries of thickness much less than the in-plane London penetration length, and at temperatures near the zero-field decoupling transition at T_* [5–7]. Specifically, we consider thin films of extreme type-II layered superconductors ($\lambda_L \to \infty$) in the intermediate regime ($H_{c1} \ll H \ll H_{c2}$) of the mixed phase, which can be described by a frustrated XY model with a finite number of N weakly coupled layers [10]. By working with the Villain form of the latter [7, 15], we obtain first that the thermodynamics factorizes into independent perpendicular and parallel parts in the presence of magnetic field at arbitrary orientation so long as the perpendicular component, $H_\perp$, of the latter is larger than the Glazman–Koshelev decoupling crossover scale [16], $B_*^\perp \sim \Phi_0/d^2\gamma^2$. The perpendicular thermodynamics is characterized by the melting of two-dimensional (2D) vortices [17] that are decoupled from the parallel Josephson vortices, as well as from the perpendicular 2D vortices in adjacent layers. This is a result of the fact that well formed vortex loops traversing a few or more layers within the film are absent in the present limit of weak inter-layer coupling [7]. In particular, the parallel Josephson vortices are unable to make connections between perpendicular 2D vortices in the same or in adjacent layer if the nearest-neighbour spacings between these perpendicular 2D vortices is much less than the zero-temperature Josephson penetration length, $\lambda_J(0) \sim \gamma d$. This situation occurs precisely for perpendicular fields that satisfy $H_\perp \gg B_*^\perp$, as stipulated above. Also, since we first take the limit of extreme type-II superconductivity, we then have the inequality $B_*^\perp \gg H_{c1}^\perp \sim \Phi_0/\lambda_L^2$. This means that the former requirement guarantees that the distance between perpendicular vortices is within the London penetration length, which in turn guarantees that magnetic screening transverse to the perpendicular field component is negligible. The parallel thermodynamics, on the other hand, is described by an LD model in parallel magnetic field $H_\parallel$, with a heavily renormalized anisotropy parameter $\gamma(T)$ that diverges exponentially as T approaches the decoupling transition temperature T_* from below. The renormalization down of the inter-layer coupling is due physically to the excitation of vortex rings [5] (fluxons) that lie in between consecutive layers.

Second, we compute the line tension of a single Josephson vortex in the simplest case of an isolated weakly coupled double-layer XY model, where we find that the parallel lower-critical field $H_{c1}^\parallel(T)$ of the double layer vanishes exponentially as T approaches T_* from below. This result agrees up to a numerical factor with a recent calculation by the author of the same critical field that employs an alternate 'frozen' superconductor description of the Meissner phase in layered superconductors [8]. Related results have also been obtained by Browne and Horovitz [18] in the setting of long Josephson junctions and by Horovitz [6] via a fermion analogy for layered superconductors in parallel magnetic field. The former coincidence is not surprising since the present double-layer type-II superconductor is equivalent to a (dynamical) long Josephson junction at zero temperature [19–22] that is effectively thinner than the London penetration length. In fact, the above analysis proceeds by first considering the length of the vortex as imaginary time. Semiclassical quantum corrections to the energy of the fundamental sine–Gordon soliton [23–25], which corresponds to the Josephson vortex in the double layer, are then computed. Entropic wandering of the vortex therefore translates into quantum fluctuations of the soliton. Note that the wandering of Josephson vortices is equivalent to the excitation of double-layer fluxons, which are again the physical origin of this phenomenon.

We next consider a one-dimensional lattice of Josephson vortices in the double layer. After adapting certain elements from the analysis of long Josephson junctions in external field [19–22] to the 'semiclassical' analysis discussed above in the case of a periodic array of sine–Gordon solitons, we are able to compute the reversible magnetization as a function of parallel magnetic field. Notably, we obtain a crossover field $B_*^{\parallel}(T) \sim \Phi_0/d^2\gamma(T)$, beyond which, generally, the magnetization displays a $B_{\parallel}^{-3}$ tail characteristic of both long Josephson junctions [19] and of layered superconductors in high parallel field [26] (see figure 1)†. Unlike long Josephson junctions, however, this crossover field is much larger than the lower-critical field. Thus we find no evidence for field dependence in the layer decoupling transition temperature, T_*, within the present 'semiclassical' approximation, which is in agreement with the results of Korshunov and Larkin [14]. Finally, we argue that such is the case for any finite number of layers, since the isolated double layer represents the weakest link.

The remainder of the paper is organized as follows: in section 2 we introduce the frustrated XY model in the Villain form, from which we derive the renormalized LD model for $T < T_*$. The double-layer case is the focal point of section 3, where we compute the parallel lower-critical field and the parallel reversible magnetization from the above LD model, in addition to the compressibility modulus of the corresponding vortex array and the effective inter-vortex interaction potential in the dilute limit. We then apply these results to the phenomenology of layered type-II superconductors in section 4, as summarized by figure 2. Finally, we assess the validity of the present 'semiclassical' approximation, as well as discuss the general case of N layers, in section 5.

2. Frustrated *XY* model

The object now is to understand extreme type-II superconducting films composed of a finite number N of weakly coupled layers in the presence of external magnetic field. In the intermediate regime of the mixed phase, where the magnetic field satisfies $H_{c1} \ll H \ll H_{c2}$, the London penetration length is, in general, much larger than the inter-vortex spacing. Following Li and Teitel [10], magnetic screening effects are then negligible, and we may describe the system by a uniformly frustrated layered XY model with an energy functional

† A recent analysis of double-layered superconductors that employs a new fermion analogy for the Lawrence–Doniach model finds that the crossover shown by the magnetization as a function of parallel magnetic field (see figure 1) is practically destroyed by entropic pressure effects for temperatures in the critical regime. See [27].

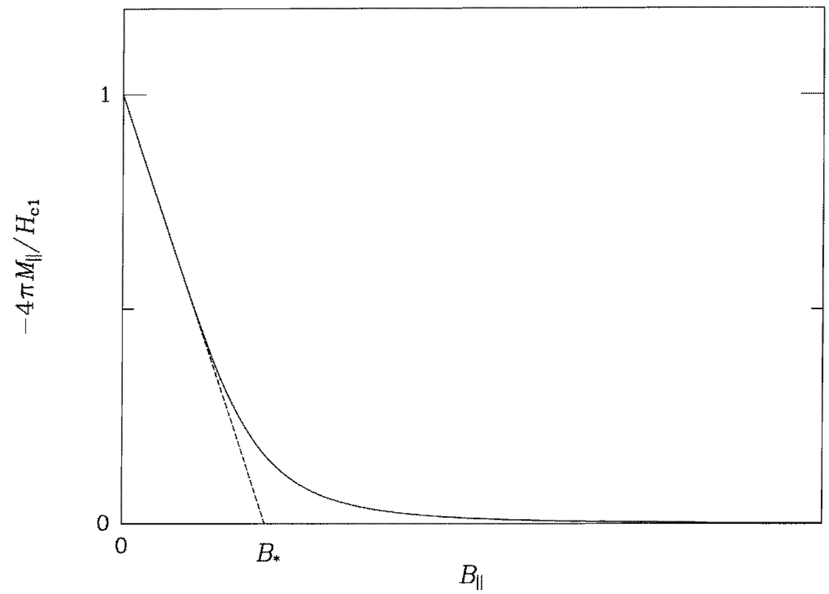

Figure 1. Shown is the reversible parallel magnetization of an extreme type-II double-layer superconductor in the intermediate regime of the mixed phase, i.e. the parallel fields satisfy $H_{c1}^{\parallel} \ll H_{\parallel} \cong B_{\parallel} \ll H_{c2}^{\parallel}$. The crossover field $B_*^{\parallel}$, in particular, satisfies the latter inequalities (see equation (51)). Also, the tail in the magnetization that appears at fields beyond this scale varies asymptotically as $H_{\parallel}^{-3}$. Although these results are generally valid only for temperatures in the Ginzburg–Landau regime, they should provide a good lower bound for $-4\pi M_{\parallel}$ at temperatures in the critical regime near T_* (see [27]).

given by

$$E_{XY} = J_{\parallel} \sum_{l=1}^{N} \sum_{\boldsymbol{r}} \sum_{\mu=x,y} \{1 - \cos[\Delta_{\mu}\phi(\boldsymbol{r}, l) - A_{\mu}(\boldsymbol{r}, l)]\}$$
$$+ J_{\perp} \sum_{l=1}^{N-1} \sum_{\boldsymbol{r}} \{1 - \cos[\phi(\boldsymbol{r}, l+1) - \phi(\boldsymbol{r}, l) - A_z(\boldsymbol{r}, l)]\}. \quad (1)$$

Here $\phi(\boldsymbol{r}, l)$ is the phase of the superconducting order parameter on the layered structure, where $\boldsymbol{r}$ ranges over the square lattice with lattice constant a, and l is the index for the layers separated by a distance d. We presume that the lattice constant a is larger than the size of a typical Cooper pair. The magnetic flux threading the plaquette at site $(\boldsymbol{r}, l)$ perpendicular to the $\mu = x, y, z$ direction reads $\Phi_{\mu} = (\Phi_0/2\pi)\sum_{\nu,\gamma} \epsilon_{\mu\nu\gamma}\Delta_{\nu}A_{\gamma}$, where $\Phi_0 = hc/2e$ is the flux quantum. Also, $\Delta_{\mu}\phi(r) = \phi(r + \hat{\mu}) - \phi(r)$ is the lattice difference operator. The nearest-neighbour Josephson couplings are related to the respective masses in Ginzburg–Landau theory by

$$J_{\parallel} = (\hbar^2/2m_{\parallel}a^2)(n_s a^2 d) \quad (2a)$$
$$J_{\perp} = (\hbar^2/2m_{\perp}d^2)(n_s a^2 d) \quad (2b)$$

where n_s labels the superfluid density. Note that $J_{\parallel}$ is independent of the lattice constant, a, as required by scale invariance in two dimensions. Hence the anisotropy parameter,

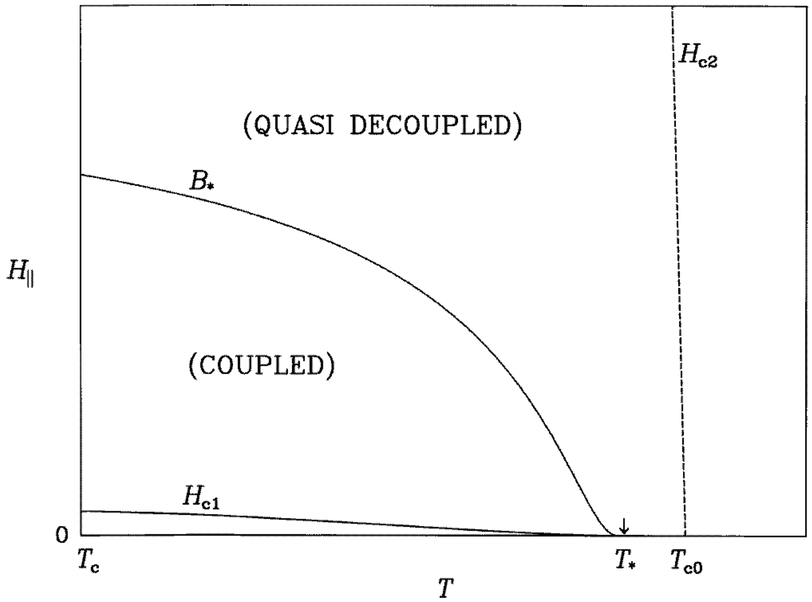

Figure 2. Shown is the phase diagram of a double-layer extreme type-II superconductor in parallel external magnetic field near the critical regime. The KT transition temperature T_c marks the point above which each individual layer becomes resistive in the absence of external magnetic field, while T_* marks the layer-decoupling transition. The non-zero slope shown by the parallel crossover field, $B_*(T)$, at temperatures below its point of inflection reflects the temperature dependence of the bare Josephson penetration scale, Λ_0 (see equation (16)). Finally, the dashed line represents the temperature profile for the parallel upper-critical field within the Ginzburg–Landau approximation.

$\gamma' = (J_\parallel/J_\perp)^{1/2}$, of the XY model is related to that of the mass, $\gamma = (m_\perp/m_\parallel)^{1/2}$, by

$$\gamma' = \gamma \frac{d}{a}. \tag{3}$$

Throughout, we will presume weak coupling between layers, $\gamma' \gg 1$.

To compute the corresponding partition function $Z = \int \mathcal{D}\phi(r)\, \mathrm{e}^{-E_{XY}/k_B T}$, we now make the usual (low-temperature) Villain substitution for the exponential factors above [15]; i.e.

$$\mathrm{e}^{-\beta(1-\cos\theta)} \to (2\pi\beta)^{-1/2} \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n\theta} \mathrm{e}^{-n^2/2\beta}.$$

After integrating over the phase-field [7], we then obtain the following dual representation equivalent to N-layered compact quantum electrodynamics (PQED) in the strong-coupling limit (modulo a constant) [28]:

$$Z = \sum_{\{n_\mu(r)\}} \Pi_r \delta\left[\sum_\nu \Delta_\nu n_\nu|_r\right] \times \exp\left[-\frac{1}{2\beta_\parallel}\sum_{l=1}^{N}\sum_{\boldsymbol{r}} \boldsymbol{n}^2(\boldsymbol{r},l) - \frac{1}{2\beta_\perp}\sum_{l=1}^{N-1}\sum_{\boldsymbol{r}} n_z^2(\boldsymbol{r},l) - \mathrm{i}\sum_{r,\nu} n_\nu(r) A_\nu(r)\right] \tag{4}$$

where $n_\mu(r)$ is an integer link field on the layered lattice structure of points $r = (\boldsymbol{r}, l)$, with $\mu = x, y, z$ and $\boldsymbol{n} = (n_x, n_y)$. Here, $\beta_{\|,\perp} = J_{\|,\perp}/k_B T$. Note that the conserved integer field n_μ is conjugate to the superfluid current $\Delta_\mu \phi - A_\mu$ of the XY model (1) in the continuum limit. To proceed further, let us now decompose the parallel field $\boldsymbol{n}$ into transverse and longitudinal parts $\boldsymbol{n}(\boldsymbol{r}, l) = \boldsymbol{n}'(\boldsymbol{r}, l) - \boldsymbol{n}_-(\boldsymbol{r}, l) + \boldsymbol{n}_-(\boldsymbol{r}, l-1)$, where the transverse and longitudinal fields, $\boldsymbol{n}'$ and $\boldsymbol{n}_-$, respectively satisfy the constraints $\boldsymbol{\nabla} \cdot \boldsymbol{n}' = 0$ and $\boldsymbol{\nabla} \cdot \boldsymbol{n}_- = n_z$, with $\boldsymbol{\nabla} = (\Delta_x, \Delta_y)$. We then take the customary potential representation $\boldsymbol{n}_- = -\boldsymbol{\nabla}\Phi$ for the longitudinal (inter-layer) field, which yields $\Phi(\boldsymbol{r}, l) = \sum_{r'} G^{(2)}(\boldsymbol{r} - \boldsymbol{r}') n_z(\boldsymbol{r}', l)$, where

$$G^{(2)}(\boldsymbol{r}) = \int_{BZ} \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}}{4 - 2\cos(k_x a) - 2\cos(k_y a)} \tag{5}$$

is (formally) the Green function for the square lattice. In the limit of weak coupling, $\gamma' \to \infty$, the interlayer field $\boldsymbol{n}_-$ vanishes, which implies that $\boldsymbol{n}'$ is indeed an integer field. After making a suitable (lattice) integration by parts of the energy functional in equation (4), we then obtain the factorization $Z = Z_{CG} \Pi_{l=1}^{N} Z_{DG}^{(l)}$ for the partition function in the limit of weakly coupled layers, where

$$Z_{DG}^{(l)} = \sum_{\{\boldsymbol{n}'(\boldsymbol{r},l)\}} \Pi_{\boldsymbol{r}} \delta[\boldsymbol{\nabla}\cdot\boldsymbol{n}'|_{\boldsymbol{r},l}] \exp\left[-\frac{1}{2\beta_\|} \sum_{\boldsymbol{r}} \boldsymbol{n}'^2(\boldsymbol{r}, l) - \mathrm{i} \sum_{\boldsymbol{r}} \boldsymbol{n}'(\boldsymbol{r}, l) \cdot \boldsymbol{A}(\boldsymbol{r}, l) \right] \tag{6}$$

is the partition function for the 2D discrete Gaussian model [29], with the in-plane vector potential $\boldsymbol{A} = (A_x, A_y)$ presumed to be in the gauge $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$, while the inter-layer Coulomb gas factor reads

$$\begin{aligned} Z_{CG} = \sum_{\{n_z(\boldsymbol{r},l)\}} \exp\Bigg\{ &-\frac{1}{2\beta_\|} \sum_{l=1}^{N} \sum_{\boldsymbol{r},\boldsymbol{r}'} [n_z(\boldsymbol{r}, l-1) - n_z(\boldsymbol{r}, l)] G^{(2)}(\boldsymbol{r} - \boldsymbol{r}') [n_z(\boldsymbol{r}', l-1) \\ &- n_z(\boldsymbol{r}', l)] - \mathrm{i} \sum_{l=1}^{N-1} \sum_{\boldsymbol{r}} n_z(\boldsymbol{r}, l) A_z(\boldsymbol{r}, l) - \frac{1}{2\beta_\perp} \sum_{l=1}^{N-1} \sum_{\boldsymbol{r}} n_z^2(\boldsymbol{r}, l) \Bigg\} \end{aligned} \tag{7}$$

with the fields at the boundary layers set to $n_z(\boldsymbol{r}, 0) = 0 = n_z(\boldsymbol{r}, N)$. This factorization is consistent with the original layered XY model (1) that consists of N independent 2D XY models in this limit. For the more relevant case of γ' large compared to one, but finite, we now make note of the fact that these XY layers remain effectively decoupled in the presence of perpendicular magnetic fields that are larger than the Glazman–Koshelev decoupling scale [16], $B_*^\perp \sim \Phi_0/\lambda_J^2(0)$, where

$$\lambda_J(0) = \gamma' a = \gamma d \tag{8}$$

is the Josephson penetration length. We therefore argue on a physical basis that the above factorization prevails in the presence of Josephson coupling as long as the perpendicular field satisfies $H_\perp \gg B_*^\perp$ in thin films of layered superconductors.

In the absence of external magnetic field, $A_\mu(r) = 0$, each layer (6) thus undergoes a KT transition at $k_B T_c \lesssim (\pi/2) J_\|$, while the inter-layer links $n_z(\boldsymbol{r})$ undergo an inverted 2D Coulomb gas binding transition at $k_B T_* = 4\pi J_\|$ in the limit of weak inter-layer coupling, $\gamma' \to \infty$ [7]. (It is understood that the limit of vanishing Josephson coupling is taken before that of vanishing perpendicular field.) The latter high-temperature transition, which occurs well inside the normal phase of each individual layer, corresponds to the decoupling of layers mediated by the binding of oppositely (n_z) charged vortex rings lying in between consecutive layers [5]. Note that this implies that Josephson coupling between *resistive* layers exists in the temperature regime $T_c < T < T_*$ [8, 11, 12]! For γ' large but finite, the form (7) of the layered Coulomb gas ensemble indicates that each set of consecutive double layers is dielectrically screened by itself as well as by the neighbouring $(N-2)$ such double layers. Hence, they each can be considered in isolation from their neighbours as long as one makes the replacement $\beta_{\parallel} \to \epsilon_0^{N-1}\beta_{\parallel}$, where $\epsilon_0 - 1$ is the polarization of an isolated double layer. Since the latter is directly proportional to the fugacity, $y_* = \exp(-2\pi\gamma'^2)$, of the Coulomb gas (7) at T_* (see equation (11), later) [8], and since the decoupling transition temperature is then given by $k_B T_* \cong 4\pi J_{\parallel}[1 + (N-1)(\epsilon_0 - 1)]$ in the limit of weak inter-layer coupling, we obtain an implicit linear dependence for the corrections to the value of T_* with the fugacity y_*. Such a linear dependence agrees with the standard renormalization-group flows that correspond to the 2D Coulomb gas [30]. As expected, the above formula for T_* also indicates that the decoupling transition temperature increases without bound with the number of layers. In particular, the former linear increase crosses over to an exponential increase at $N_0 \sim (\epsilon_0 - 1)^{-1}$ layers, which is far beyond the 2D–3D crossover point expected to occur at $N_{c/0} \sim \gamma'$ layers for the layered XY model [31]. This is then consistent with the fact that the *bulk* layered XY model exhibits only a $3D$ superfluid transition at the bulk T_c. Note that the present factorization into parallel and perpendicular parts is unable to obtain corrections for the value of T_c in the case of large but finite anisotropy parameters γ' [32].

Consider now equations (6) and (7) in the presence of a homogeneous magnetic induction,

$$B_{\parallel} = \frac{\Phi_0}{2\pi d} b_{\parallel} \tag{9a}$$

$$B_{\perp} = \frac{\Phi_0}{2\pi a} b_{\perp} \tag{9b}$$

with the parallel component directed along the y-axis, and with $B_{\perp} \gg B_*^{\perp}$ to insure the decoupling between 2D perpendicular vortices and parallel Josephson vortices explicit in the previous factorization of (4) [16]. This decoupling becomes evident if we choose the gauge $A_x = 0$, $A_y = b_{\perp}x$, and $A_z = -b_{\parallel}x$, where $b_{\parallel}$ and $b_{\perp}$ are the parallel (7) and perpendicular (6) magnetic flux densities, respectively. In particular, each layer independently experiences the perpendicular component $B_{\perp} = H_{\perp}$ of the magnetic induction†, which sets the intra-layer vortex density to be $n_V = |H_{\perp}|/\Phi_0$. The fact that $n_V \gg \lambda_J^{-2} \gg \lambda_L^{-2}$ in the present limit of extreme type-II superconductivity insures that magnetic screening effects transverse to the perpendicular field component can be neglected. Each layer will then independently follow the 2D vortex lattice melting scenario [17], with a melting temperature $T_m < T_c$. Since the issue of 2D vortex lattice melting has already been discussed extensively in the literature with respect to the phenomenon of high-temperature superconductivity [1], we shall end the present discussion here and focus our attention below on the thermodynamics connected with the parallel component to the magnetic induction.

We now derive the renormalized LD theory [2–4] mentioned in the introduction. It is useful first to make the following Hubbard–Stratonovich transformation of the Coulomb gas ensemble (7) [34]:

$$Z_{CG} = \int \mathcal{D}\theta(\boldsymbol{r}, l) \sum_{\{n_z(\boldsymbol{r},l)\}} \exp\biggl\{ -\frac{\beta_{\parallel}}{2} \sum_{l=1}^{N} \sum_{\boldsymbol{r}} (\boldsymbol{\nabla}\theta)^2 - \mathrm{i} \sum_{l=1}^{N} \sum_{\boldsymbol{r}} \theta(\boldsymbol{r}, l)[n_z(\boldsymbol{r}, l) - n_z(\boldsymbol{r}, l-1)] - \mathrm{i} \sum_{l=1}^{N-1} \sum_{\boldsymbol{r}} n_z(\boldsymbol{r}, l) A_z(\boldsymbol{r}, l) - \frac{1}{2\beta_{\perp}} \sum_{l=1}^{N-1} \sum_{\boldsymbol{r}} n_z^2(\boldsymbol{r}, l) \biggr\}. \tag{10}$$

† The perpendicular 2D vortices become coherent perpendicular vortex lines that are widened by parallel excursions (Josephson vortices) in the low-field regime $B_{\perp} \ll B_*^{\perp}$. See [16] and [33].

Here $\theta(\boldsymbol{r}, l)$ represents a real scalar field that lives on each layer. Suppose now that we operate in the low-temperature regime, $T < T_*$, where the layers are coupled [5–7]. Then inter-layer n_z charges in the Coulomb-gas ensemble are screened, which means that global charge conservation is no longer enforced. Following the standard prescription [34], we independently sum over charge configurations at each site, with the restriction to values $n_z = 0, \pm 1$. In the limit that the fugacity

$$y_0 = \exp(-1/2\beta_\perp) \tag{11}$$

is small, we then find that the original Coulomb gas ensemble (7) is equivalent to a renormalized Lawrence–Doniach model $Z_{CG} = \int \mathcal{D}\theta(\boldsymbol{r}, l) \exp(-E_{LD}/k_B T)$ with energy functional

$$E_{LD} = J_\parallel \sum_{l=1}^{N} \sum_{\boldsymbol{r}} \frac{1}{2} (\nabla\theta)^2 + 2y_0 k_B T \sum_{l=1}^{N-1} \sum_{\boldsymbol{r}} \{1 - \cos[\theta(\boldsymbol{r}, l+1) - \theta(\boldsymbol{r}, l) - A_z(\boldsymbol{r}, l)]\}. \tag{12}$$

At the decoupling transition in particular, we have that the fugacity (11) is given by $y_* = \exp(-2\pi\gamma'^2)$. Hence, the anisotropy parameter is renormalized up to

$$\gamma'_* = \left(\frac{J_\parallel}{2y_* k_B T_*}\right)^{1/2} = (8\pi)^{-1/2} \mathrm{e}^{\pi\gamma'^2} \tag{13}$$

at the decoupling transition in the present LD model. Since the n_z charges physically represent vortex rings (fluxons) that lie in between consecutive layers [5], we conclude that such excitations are responsible for the renormalization (13) of the mass anisotropy near the decoupling transition. In closing, we remind the reader that the above derivation of model (12) is valid only for fugacities (11) that satisfy $y_0 \ll 1$, i.e. for temperatures $T > J_\perp/k_B$.

3. Double layer

We shall now consider the parallel thermodynamics associated with the renormalized LD model (12) in the special case of a double layer ($N = 2$), which is analytically tractable. This case is very similar to a long Josephson junction [19–22] restricted to pass no net current between the junction. Although the vortex lattice that results from the infinite-layer LD model in a parallel field has far more structure than the simple vortex array corresponding to an isolated double layer [26], we believe that it is sufficient to study the latter with respect to the issue of layer decoupling in general, since it represents the weakest link.

For the special case of two weakly coupled layers in the presence of a homogeneous parallel magnetic induction $B_\parallel$ directed along the y-axis, the partition function corresponding to the renormalized LD model energy functional (12) is expressible as

$$Z_{CG} = \int \mathcal{D}\bar{\theta}(\boldsymbol{r})\mathcal{D}\phi(\boldsymbol{r}') \exp\left\{-\bar{\hbar}_F^{-1} \int \mathrm{d}y\, L_{SG}[\phi] - \int \mathrm{d}^2 r \frac{\beta_\parallel}{2} (\nabla\bar{\theta})^2\right\} \tag{14}$$

where $\bar{\theta}(\boldsymbol{r}) = 2^{-1/2}[\theta(\boldsymbol{r}, 1) + \theta(\boldsymbol{r}, 2)]$. Here,

$$L_{SG}[\phi] = \int \mathrm{d}x \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x} - b_\parallel\right)^2 + \Lambda_0^{-2}(1 - \cos\phi)\right] \tag{15}$$

represents the 'Lagrangian' for the sine–Gordon model in one space (x) and one imaginary time ($y = \mathrm{i}\bar{t}$) dimension, with a bare temperature-dependent Josephson penetration length

$$\Lambda_0 = a(\beta_\parallel/4y_0)^{1/2} \tag{16}$$

while the effective dimensionless Planck constant is

$$\bar{\bar{h}}_F = 2/\beta_\parallel. \tag{17}$$

Notice that we have taken the continuum limit of the LD model (12), as well as made the change of variable $\phi(\boldsymbol{r}) = \theta(\boldsymbol{r}, 2) - \theta(\boldsymbol{r}, 1) - A_z(\boldsymbol{r})$. The integration over $\bar{\theta}$ on the right of equation (14) results in a trivial Gaussian factor. Below, we shall exploit the quantum mechanical analogy suggested above for the non-trivial sine–Gordon factor in order to compute the parallel lower-critical field [8] and the parallel reversible magnetization of the double-layer system.

3.1. Single Josephson vortex

We now set ourselves to the task of computing the parallel lower-critical field, $H_{c1}^\parallel$, of the double-layer system, which is in general related to the free energy per unit length of a single Josephson vortex, $\varepsilon_\parallel$, by [1] $H_{c1}^\parallel = 4\pi\varepsilon_\parallel/\Phi_0$. Let us therefore consider the effective sine–Gordon model (15) in the presence of a single Josephson vortex aligned along the y-axis, i.e. the homogeneous magnetic flux is set to $b_\parallel = 0$, while the phase-difference field winds once along the x-axis; $\int_{-\infty}^{\infty} \mathrm{d}x\, \partial\phi/\partial x = 2\pi$. In the absence of thermal (or 'quantum') fluctuations, the vortex line tension is given by the Ginzburg–Landau energy

$$\varepsilon_\parallel^0 = \frac{J_\parallel}{2} L_{SG}[\phi_0] = \frac{4J_\parallel}{\Lambda_0} \tag{18}$$

of the 'static' fundamental sine–Gordon soliton

$$\phi_0(x, y) = 4\tan^{-1} \mathrm{e}^{x/\Lambda_0} \tag{19}$$

which is a solution of the field equation

$$-\frac{\partial^2\phi}{\partial x^2} + \Lambda_0^{-2}\sin\phi = 0 \tag{20}$$

obtained by minimizing the corresponding 'action' $L_y L_{SG}[\phi_0]$ [23, 25], and which represents the single Josephson vortex. For temperatures near the decoupling transition at T_*, however, vortex wandering is critical [8], and entropic (or 'quantum mechanical') corrections to the vortex line energy (18) must be accounted for.

We shall now include the effects of thermal wandering in the double-layer Josephson vortex (19) by first Wick rotating the y coordinate to a real time-like coordinate, $y = \mathrm{i}\bar{t}$. Second, we observe that *the free energy per unit length of the Josephson vortex is equal to the product of* $\frac{1}{2}J_\parallel$ *with the 'quantum mechanically' renormalized 'mass' of the fundamental sine–Gordon soliton*. To obtain the latter, we shall employ the 'semiclassical' approximation [24, 25] generally valid in the limit $\bar{\bar{h}}_F \to 0$, i.e. at low temperature. In particular, consider small deviations $\phi = \phi_0 + \phi_1$ from the 'static' vortex configuration (19). Then integration by parts yields that (15) is approximately

$$L_{SG}[\phi] \cong L_{SG}[\phi_0] + \frac{1}{2}\int \mathrm{d}x\, \phi_1^* \left(\frac{\partial^2}{\partial \bar{t}^2} - \frac{\partial^2}{\partial x^2} + \Lambda_0^{-2}\cos\phi_0\right)\phi_1 \tag{21}$$

to second order in the deviation. Hence, in the presence of the soliton, we obtain a spectrum of harmonic oscillators of the form $\phi_1(x, \bar{t}) = \psi(x)\mathrm{e}^{\mathrm{i}\omega\bar{t}}$, where

$$\left(-\frac{\partial^2}{\partial x^2} + \Lambda_0^{-2}\cos\phi_0\right)\psi = \omega^2\psi. \tag{22}$$

It is well known [23, 25] that the spectrum corresponding to (22) is composed of a zero mode

$$\psi_b(x) = \mathrm{sech}\,(x/\Lambda_0) \qquad \omega_b = 0 \tag{23}$$

that lies within the gap of the continuum

$$\psi_k(x) = \mathrm{e}^{\mathrm{i}kx}[k + \mathrm{i}\Lambda_0^{-1}\tanh(x/\Lambda_0)] \qquad \omega_k = (k^2 + \Lambda_0^{-2})^{1/2}. \tag{24}$$

In the 'semiclassical' approximation [25], the 'transition amplitude' over a period of 'time' $\bar{T}_0$ is given by the product $Z_{SG}[1] = \exp(-\mathrm{i}\bar{T}_0\varepsilon_{\parallel}^0/k_BT)\Pi_k z_k$, where

$$z_k = \sum_{n=0}^{\infty} \exp\left[-\mathrm{i}\omega_k\bar{T}_0\left(n + \frac{1}{2}\right)\right] \tag{25}$$

gives the corresponding amplitude of each harmonic oscillator. After Wick rotating back to imaginary time $L_y = \mathrm{i}\bar{T}_0$, we observe that only the $n = 0$ terms above survive the limit of a long vortex, $L_y \to \infty$. Yet the ratio of the partition functions in the presence of a single Josephson vortex to that in its absence is in general related to the vortex line tension, $\varepsilon_{\parallel}$, by $Z_{SG}[1]/Z_{SG}[0] = \exp(-L_y\varepsilon_{\parallel}/k_BT)$. In the 'semiclassical' limit, therefore, we obtain

$$\varepsilon_{\parallel} = \varepsilon_{\parallel}^0 + \frac{k_BT}{2}\left(\sum_k \omega_k - \sum_q \omega_q\right) \tag{26}$$

for the line tension [24, 25], where the k sum and the q sum above correspond respectively to traces of the zero-point energy in the presence and in the absence of the fundamental sine–Gordon soliton. In particular, the presumption of periodic boundary conditions along x implies the quantization conditions $kL_x + \delta(k) = 2\pi n$ and $qL_x = 2\pi n$, where

$$\delta(k) = 2\tan^{-1}(1/k\Lambda_0) \tag{27}$$

is the phase shift of the continuum states (24), and where n is any integer. Properly counting these states then yields that the difference in brackets, $\sum_n(\omega_k - \omega_q)$, in equation (26) is equal to [24, 25] $-(2\pi)^{-1}\int_{-\infty}^{\infty} \mathrm{d}k(\mathrm{d}\omega_k/\mathrm{d}k)\delta(k)$ in the limit $L_x \to \infty$. After introducing a momentum cut-off, r_0^{-1}, and integrating by parts, we obtain

$$\varepsilon_{\parallel} = \varepsilon_{\parallel}^0[1 - (4\pi\beta_{\parallel})^{-1}\ln(\Lambda_0/r_0)]. \tag{28}$$

Notice that the correction above to the low-temperature line tension is first order in $\bar{\hbar}_F$, and that it can be interpreted as a renormalization to the 'mass', Λ_0^{-1}, of the sine–Gordon model (15). Since a renormalization group exists, we may now express the line tension as

$$\varepsilon_{\parallel} = 4J_{\parallel}/\lambda_J \tag{29}$$

and then iterate equation (28), which yields $\lambda_J^{-1} = \Lambda_0^{-1}\exp[-(4\pi\beta_{\parallel}\epsilon_0)^{-1}\ln(\lambda_J/r_0)]$, or

$$\frac{r_0}{\lambda_J} = \left(\frac{r_0}{\Lambda_0}\right)^{[1-(4\pi\beta_{\parallel}\epsilon_0)^{-1}]^{-1}}. \tag{30}$$

Above, we have replaced the right-hand side of equation (28) by the previous exponential and included the dielectric correction ϵ_0 to the on-site Coulomb-gas (7) potential, $\ln(\lambda_J/r_0)$. Employing the standard renormalization group result [30], $\epsilon_0 = 1 + \mathrm{O}[(\beta_{\parallel} - \beta_*)^{1/2}]$, for the dielectric constant of the 2D Coulomb gas at (inverted) temperatures $\beta_{\parallel}$ just above the (inverted) transition temperature $\beta_* = 1/4\pi$, we thus obtain that the renormalized Josephson penetration length λ_J diverges exponentially as it approaches the decoupling transition like

$$\lambda_J/a = C\exp[D/(\beta_{\parallel} - \beta_*)^{1/2}]. \tag{31}$$

Here, C and D are non-universal numerical constants.

In conclusion, the parallel lower-critical field $H_{c1}^{\parallel} = 4\pi\varepsilon_{\parallel}/\Phi_0$ vanishes exponentially fast near the decoupling transition of the double layer following equations (29) and (31). Horovitz has obtained this result employing a fermion analogy for layered superconductors in parallel magnetic field [6]. A similar dependence has also been proposed by Browne and Horovitz for the lower-critical field of long Josephson junctions [18]. Combining equation (2a) with the identity $(\Phi_0/\lambda_L)^2 = (2\pi)^3(\hbar^2/2m_\parallel)n_s$ for the bulk ($N \to \infty$) in-plane London penetration length λ_L yields

$$\frac{J_\parallel}{d} = (2\pi)^{-3}\frac{\Phi_0^2}{\lambda_L^2} \tag{32}$$

from which we obtain the useful expression

$$\varepsilon_\parallel(T) = \frac{8}{\pi}\gamma^{-1}\left[\frac{\Phi_0}{4\pi\lambda_L(T)}\right]^2\frac{\lambda_J(0)}{\lambda_J(T)}. \tag{33}$$

Given that $C \sim \gamma'$ in equation (31), which is consistent with expression (8) for the Josephson penetration length of the original XY model (1) at zero temperature, then we have by equation (31) that $\lambda_J(0)/\lambda_J(T) \sim \exp[-D/(\beta_\parallel - \beta_*)^{1/2}]$ near the decoupling transition. This result agrees up to a numerical constant with a previous calculation by the author of the same quantity using an alternative 'frozen' superconductor model for highly anisotropic extreme type-II superconductors in the Meissner phase [8]. Notice then that expression (33) for the parallel line tension is essentially independent of the lattice constant a, as expected by the 2D scale invariance of the LD model (1) for $T \geqslant T_*$.

Before we go on to consider an array of Josephson vortices in the next section, a few remarks are called for. First, note that the point at which the soliton 'mass', $2\varepsilon_\parallel/J_\parallel$, vanishes coincides with the layer decoupling transition. Similar effects are found in the case of the nonlinear σ model in two space and one time dimensions, which describes the quantum 2D antiferromagnet. In particular, the quantum mechanically renormalized energy of the corresponding topological soliton called a skyrmion vanishes precisely at the zero-temperature quantum critical phase transition into the quantum disordered phase characterized by a spin gap [35]. Second, note that we have essentially recovered the standard renormalization group results for the KT transition [30] via the present semiclassical quantization of the sine–Gordon soliton energy in one space and one time dimension. Finally, also observe that the entropic correction to the line-tension in equation (28) indicates that the number of microstates per unit length a of a Josephson vortex in thermal equilibrium is λ_J/r_0. Given that $r_0 \sim a$, then λ_J can be naturally interpreted as the effective width of the Josephson vortex.

3.2. Array of Josephson vortices

Consider now the case of a non-zero homogeneous magnetic induction aligned parallel to the y-axis of the double layer, i.e. $b_\parallel \neq 0$. Then it is easily seen from equation (15) that the superfluid portion,

$$G_s - G_n = -k_B T \ln\int \mathcal{D}\phi \exp\left[-\bar{\hbar}_F^{-1}\int \mathrm{d}y L_{SG}[\phi]\right] \tag{34}$$

of the Gibbs free energy [36] is minimized with respect to $b_\parallel$ at $b_\parallel = L_x^{-1}\int_{-\infty}^{\infty}\partial\phi/\partial x$. In other words, the average winding per unit length in any configuration of the phase difference between the double layers is set by the magnetic induction. In the particular case of the low-temperature 'classical' configuration, we then have that the parallel magnetic induction is related to the lattice constant a_0 of the corresponding array of Josephson vortices by

$$b_\parallel = \frac{2\pi}{a_0}. \tag{35}$$

Clearly, we expect qualitative differences between the thermodynamics of the low-field regime, $a_0 \gg \lambda_J$, and of the high-field regime, $a_0 \ll \lambda_J$. Yet does a decoupling phase transition separate the two regions, as has been suggested in the literature [3]? Below we give evidence for the existence of only a crossover [14] on the basis of a 'semiclassical' analysis of the reversible magnetization (see [27]) and of the elastic compression modulus of the vortex array.

We now set ourselves to the task of computing the parallel reversible magnetization [36]

$$M_\parallel \cong -\frac{\partial}{\partial B_\parallel}\left(\frac{G_s - G_n}{L_x L_y d}\right) \tag{36}$$

of the double layer in the intermediate regime of the mixed phase, $H_{c1}^\parallel \ll B_\parallel \ll H_{c2}^\parallel$, where $H_\parallel \cong B_\parallel$. Consider first the lowest-order Ginzburg–Landau contribution

$$G_s^0 - G_n = L_y \frac{J_\parallel}{2} L_{SG}[\phi_0] \tag{37}$$

to the Gibbs free energy in powers of $\bar{\bar{h}}_F$, where $\phi_0(x)$ represents the whirling pendulum solution of field equation (20) with spatial period a_0, i.e. [19–22]

$$\frac{d\phi_0}{dx} = \frac{2}{\kappa\Lambda_0}\mathrm{dn}\left(\frac{x - x_0}{\kappa\Lambda_0}\middle|\kappa^2\right) \tag{38}$$

where the parameter κ lies in the interval between zero and unity, and is set by the period a_0 following

$$a_0 = 2\Lambda_0 \kappa K(\kappa^2). \tag{39}$$

Above, $\mathrm{dn}(u|\kappa^2)$ represents the appropriate Jacobian elliptic function, while $K(\kappa^2)$ represents the complete elliptic integral of the first kind [37]. Conservation of energy, $E_0 = 2\Lambda_0^{-2}(\kappa^{-2} - 1)$, in the corresponding pendulum system yields

$$\frac{L_{SG}[\phi_0]}{L_x} = 2\left[a_0^{-1}\int_0^{a_0} dx \frac{1}{2}\left(\frac{d\phi_0}{dx}\right)^2\right] - E_0 - \frac{1}{2}b_\parallel^2.$$

We obtain, therefore, that the zero-order Gibbs free energy density is equal to

$$\frac{G_s^0 - G_n}{L_x L_y} = \frac{J_\parallel}{\Lambda_0^2}\left[\frac{2}{\kappa^2}\frac{E(\kappa^2)}{K(\kappa^2)} + 1 - \kappa^{-2}\right] - \frac{J_\parallel}{2}\frac{b_\parallel^2}{2} \tag{40}$$

where $E(\kappa^2)$ represents the complete elliptic integral of the second kind [37]. Standard manipulations [19] then yield that the reversible magnetization (36) is given by

$$-4\pi M_\parallel = H_{c1}^\parallel\left[\frac{E(\kappa^2)}{\kappa} - \frac{\pi^2}{4}\frac{1}{\kappa K(\kappa^2)}\right] \tag{41}$$

where $H_{c1}^\parallel = 4\pi\varepsilon_\parallel^0/\Phi_0$ is the parallel lower-critical field in the Ginzburg–Landau theory approximation. Note that $H_{c1}^\parallel$ naturally sets the maximum value of the diamagnetic magnetization (41) at zero magnetic induction ($\kappa = 1$). In particular, at low magnetic inductions $a_0 \gg \Lambda_0$, we have by (39) that $\kappa^2 \cong 1 - 16\,e^{-a_0/\Lambda_0}$. We then obtain the limiting behaviour

$$-4\pi M_\parallel \cong H_{c1}^\parallel\left[1 + 4\,e^{-a_0/\Lambda_0}\left(\frac{a_0}{\Lambda_0} + 1\right) - \frac{\pi^2}{2}\frac{\Lambda_0}{a_0}\right] \tag{42}$$

for the reversible magnetization (41). Hence, the low-field magnetization extrapolates to zero at

$$B_0^{\parallel} = \frac{2}{\pi^2}\frac{\Phi_0}{\Lambda_0 d} \tag{43}$$

which defines a bare crossover field. At high fields $a_0 \ll \Lambda_0$, on the other hand, (39) dictates that $\kappa \cong \pi^{-1} a_0/\Lambda_0$. We then obtain that the limiting behaviour for the reversible magnetization (41) is given by

$$-4\pi M_{\parallel} \cong \frac{H_{c1}^{\parallel}}{64\pi^2}\left(\frac{a_0}{\Lambda_0}\right)^3 \tag{44}$$

in such a case. This implies a $B_{\parallel}^{-3}$ tail at high fields $B_{\parallel} \gg B_0^{\parallel}$ which is characteristic of long Josephson junctions and of layered superconductors in general [19, 26]. A plot of result (41) spanning both the high-field and low-field limits is shown in figure 1. Note that the bare crossover field $B_0^{\parallel}$ is much larger than the Ginzburg–Landau lower-critical field $H_{c1}^{\parallel}$ in the present double-layer extreme type-II superconductor (see equation (51), later). This is qualitatively different from the case of a long Josephson junction [19], where $B_0^{\parallel} \sim H_{c1}^{\parallel}$.

In analogy with the previous analysis of a single Josephson vortex, let us now consider the effect of 'semiclassical' corrections to the reversible magnetization in a parallel field (41). We again have a spectrum (21) of harmonic oscillators $\phi_1(x,\bar{t}) = \psi(x)\,\mathrm{e}^{\mathrm{i}\omega\bar{t}}$ that satisfy the linearized field equation (22), but with a periodic configuration for the phase difference set by [20, 21]

$$\cos\frac{1}{2}\phi_0 = -\mathrm{sn}\left(\frac{x-x_0}{\kappa\Lambda_0}\middle|\kappa^2\right) \tag{45}$$

where $\mathrm{sn}(u|\kappa^2)$ represents the appropriate Jacobian elliptic function [37]. To be more specific, the spatial factors of each oscillator satisfy Lamé's equation [21],

$$\left[-\frac{\partial^2}{\partial x^2} + 2\Lambda_0^{-2}\mathrm{sn}^2\left(\frac{x-x_0}{\kappa\Lambda_0}\middle|\kappa^2\right) - \Lambda_0^{-2}\right]\psi = \omega^2\psi. \tag{46}$$

To make contact with the previous discussion of a single Josephson vortex, let us now focus our attention on the (bare) low-field regime $B_{\parallel} \ll B_0^{\parallel}$, where the parameter κ is exponentially close to unity, since $a_0 \gg \Lambda_0$. This allows us to approximate the potential terms in Lamé's equation $[-\partial^2/\partial x^2 + V(x)]\psi = (\omega^2 - \Lambda_0^{-2})\psi$ by

$$V(x) \cong -\sum_{n=-\infty}^{\infty} 2\Lambda_0^{-2}\,\mathrm{sech}^2\left(\frac{x-x_0-na_0}{\Lambda_0}\right) \tag{47}$$

where each term above corresponds to the potential associated with a fundamental sine–Gordon soliton centred at $x_0 + na_0$. In general, the band structure corresponding to Lamé's equation (46) is composed of a continuum and a zero-mode band separated by a gap [21, 37]. A curious feature particular to each potential term in equation (47), however, is its *transparency* [23], i.e. the continuum oscillators (24) of the fundamental sine–Gordon soliton have no reflected wave component. Therefore, in the present (bare) low-field limit, the upper continuum band is essentially the same as that of a fundamental soliton (24). Repeating the renormalization group arguments made in the previous section for the line energy of a single Josephson vortex then indicates that the entropic correction due to the latter continuum band can be accounted for by simply replacing Λ_0 (16) with λ_J (31) in the original Ginzburg–Landau free energy of the vortex lattice, i.e.

$$G_s - G_n = \frac{J_{\parallel}}{2}\int \mathrm{d}y \int \mathrm{d}x \left[\frac{1}{2}\left(\frac{\partial\phi_0}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi_0}{\partial x}\right)^2 - \frac{1}{2}b_{\parallel}^2 + \lambda_J^{-2}(1-\cos\phi_0)\right] \tag{48}$$

with the lattice constant of the vortex array (38) set by equation (35). In general, however, the effects of the zero-mode band must also be included in the present semiclassical analysis. The corresponding states are given by the tight-binding ansatz $|k_0\rangle = \sum_n \mathrm{e}^{\mathrm{i}k_0 a_0 n}|n\rangle$ in the present (bare) low-field limit, where $\langle x|n\rangle = \psi_b(x - x_0 - na_0)$ is the (normalized) bound state (23) of the fundamental soliton located at the nth well. The hopping matrix element is therefore

$$-t_0 = \langle n| - \frac{\partial^2}{\partial x^2} + V(x)|n+1\rangle = (\omega^2 - \Lambda_0^{-2})\langle n|n+1\rangle.$$

But $\langle n|n+1\rangle \cong 2\pi\,\mathrm{e}^{-a_0/\Lambda_0}$ is much less than unity, which yields $t_0 \cong (2\pi/\Lambda_0^2)\,\mathrm{e}^{-a_0/\Lambda_0}$. This means that the zero-mode band has a spectrum $\omega_0(k_0) = 2t_0^{1/2}|\sin(\frac{1}{2}k_0 a_0)|$ that is exponentially narrow. By (25), the zero-mode band results in an entropic *pressure* contribution to the Gibbs free energy density given by

$$P_0 = \frac{k_B T}{d} L_x^{-1} \sum_{k_0} \frac{1}{2}\omega_0(k_0) = \frac{2}{\pi}\frac{k_B T}{a_0 d} t_0^{1/2}. \tag{49}$$

Hence, the magnetization (36) acquires a *diamagnetic* correction $-\partial P_0/\partial B_\parallel$ of order $\mathrm{e}^{-a_0/2\Lambda_0}$, which is negligibly small in the present (bare) low-field limit. Equation (42) indicates, however, that the low-field correction to the initial linear increase of the parallel magnetization varies as $\mathrm{e}^{-a_0/\Lambda_0}$ in the Ginzburg–Landau regime. Unlike the case of a single Josephson vortex, then no obvious renormalization group appears to exist for the above entropic pressure contribution.

In conclusion, double-layer extreme type-II superconductors in parallel magnetic field are described by the effective Ginzburg–Landau free energy (48), along with the boundary condition (35), in the bare low-field limit $B_\parallel \ll B_0^\parallel$ of the intermediate regime, $H_{c1}^\parallel \ll B_\parallel \ll H_{c2}^\parallel$. This means that the reversible magnetization is determined by the original Ginzburg–Landau theory analysis (equations (36)–(44)), where the bare Josephson penetration length Λ_0 is replaced by the renormalized length λ_J throughout. In particular, the true parallel crossover field (see figure 1) of the double layer is given by

$$B_*^\parallel = \frac{2}{\pi^2}\frac{\Phi_0}{\lambda_J d} \tag{50}$$

instead of by equation (43). However, our inability to find a renormalization group for the entropic pressure contribution (49) to the parallel magnetization suggests that the present renormalized Ginzburg–Landau theory result for $-4\pi M_\parallel$ serves only as a strong lower bound in the critical regime [27]. We therefore find evidence for at best a crossover as a function of magnetic field below the bare scale $B_0^\parallel$, and no evidence for a decoupling phase transition at fixed temperature. Finally, it is easily shown after employing relation (32) that

$$\frac{B_*^\parallel}{H_{c1}^\parallel} = \frac{\lambda_L^2}{d^2}. \tag{51}$$

This of course indicates that the crossover field is much larger than the lower-critical field, which validates *a posteriori* the assumption (36) that $H_\parallel \cong B_\parallel$ in the intermediate regime of the mixed phase. It also illustrates the qualitative difference between a double-layer superconductor and a long Josephson junction [19], where $B_*^\parallel \sim H_{c1}^\parallel$.

We shall close this section by computing the compression modulus of the parallel array of Josephson vortices, as well as the interaction energy between widely spaced vortices.

The local change in the elastic free-energy density due to a local fluctuation δn_V in the vortex density is given by $\delta f_{SG} = \frac{1}{2}(\partial^2 f_{SG}/\partial n_V^2)(\delta n_V)^2$, where

$$f_{SG} = \frac{J_\parallel}{\lambda_J^2 d}\left[\frac{2}{\kappa^2}\frac{E(\kappa^2)}{K(\kappa^2)} + 1 - \kappa^{-2}\right] \tag{52}$$

is the Gibbs free-energy density (48) modulo the $-\frac{1}{2}b_\parallel^2$ term (see equation (40)), which can be considered as part of the magnetic field energy. In general, the number of Josephson vortices per unit length along the x-axis is $n_V = a_0^{-1} = b_\parallel/2\pi$. By differentiating the first term in equation (41) once with respect to $b_\parallel$, we thus obtain $\partial^2 f_{SG}/\partial n_V^2 = (8J_\parallel/d)(1-\kappa^2)[K(\kappa^2)]^3/E(\kappa^2)$. Now the displacement field $u(x)$ of the vortex array along the x-axis is related to the density fluctuation by $\delta n_V = -a_0^{-1}(\partial u/\partial x)$. Therefore, employing previous identities ((9a), (32) and (35)), we find that the elastic energy is given by $\delta f_{SG} = \frac{1}{2}c_{11}(\partial u/\partial x)^2$, where the compression modulus reads

$$c_{11} = \pi^{-3}\frac{d^2 B_\parallel^2}{\lambda_L^2}\frac{[K(\kappa^2)]^3}{E(\kappa^2)}(1-\kappa^2). \tag{53}$$

In the low-field limit $B_\parallel \ll B_*^\parallel$ we then have that the array is exponentially soft [22], with $c_{11} \propto \mathrm{e}^{-a_0/\lambda_J}$. Similar softening of the Abrikosov vortex lattice occurs in conventional superconductors near the lower critical field, but with the Josephson penetration length λ_J replaced by the London penetration length λ_L [38]. On the other hand, the high-field limit $B_\parallel \gg B_*^\parallel$ yields that $c_{11} \cong (4\pi)^{-1}(d/\lambda_L)^2 B_\parallel^2$, which is a formula characteristic of the elastic moduli in extreme type-II superconductors ($\lambda_L \to \infty$) generally [1].

Finally, we may define the interaction energy between two well separated vortices by subtracting the line tension (29) from the free energy (52) per unit length of a single period in the array of Josephson vortices, i.e. the repulsive interaction energy per unit length is given by

$$v(a_0) = d a_0 f_{SG} - \varepsilon_\parallel \tag{54}$$

in the low-field limit $a_0 \gg \lambda_J$, which after some analysis yields $v(a_0) = J_\parallel \lambda_J^{-1}(1-\kappa^2)$, or

$$v(x) = 16 J_\parallel \lambda_J^{-1}\mathrm{e}^{-|x|/\lambda_J}. \tag{55}$$

Note that the Josephson penetration length λ_J acts as the screening length for the interaction between Josephson vortices instead of the London penetration length, which plays the same role in conventional type-II superconductors.

4. Phenomenology

We shall now examine the phenomenological consequences of the previous results for (double) layered superconductivity in parallel external magnetic field. Let us first consider the critical properties of the decoupling transition in the the absence of parallel magnetic induction, i.e. take $H_\parallel$ near $H_{c1}^\parallel$. Given the standard Ginzburg–Landau dependence, $\lambda_L(T) = \lambda_0(1 - T/T_{c0})^{-1/2}$, for the bulk ($N \to \infty$) in-plane London penetration length, then (32) yields

$$J_\parallel(T) = \gamma^{-1} k_B T_0 (1 - T/T_{c0}) \tag{56}$$

for the intra-layer Josephson coupling energy, where

$$k_B T_0 = \frac{2}{\pi}\left(\frac{\Phi_0}{4\pi\lambda_0}\right)^2 \gamma d \tag{57}$$

is the basic energy scale of the problem. Since the zero-field decoupling transition occurs at $k_B T_* = 4\pi J_\parallel(T_*)$, we then obtain $T_* = [T_{c0}^{-1} + \gamma(4\pi T_0)^{-1}]^{-1}$ for the decoupling transition temperature. This means that the size of the critical regime is

$$\delta T_* = T_{c0} - T_* \cong \gamma T_{c0}^2 / 4\pi T_0 \tag{58}$$

for $T_{c0} \ll T_0$, which is typical. Likewise, the critical temperature T_c at which each individual layer undergoes a superfluid KT transition is set by $k_B T_c \cong (\pi/2) J_\parallel(T_c)$, or $T_c \cong [T_{c0}^{-1} + \gamma((\pi/2)T_0)^{-1}]^{-1}$. Hence, the distance of this intra-layer resistive transition to the Ginzburg–Landau transition temperature T_{c0} is $\delta T_c = T_{c0} - T_c \sim 10\,\delta T_*$, as indicated by figure 2. Again, we highlight the extraordinary regime in temperature $T_c < T < T_*$ where the layers are normal yet Josephson coupled, i.e. $\rho_\parallel \neq 0$ and $H_{c1}^\parallel(T) \neq 0$. This effect has been recently observed in resistance measurements on the highly anisotropic bismuth-based series of high-temperature superconductors [11, 12].

Results similar to those outlined above have been obtained recently by the author using an alternative anisotropic 'frozen' superconductor model for the Meissner phase [8], but with the important exception that the zero-temperature Josephson penetration length (8) appearing above in equation (57) is replaced therein by the zero-temperature London penetration length λ_0. This discrepancy can be understood as follows: in the present frustrated XY model description (1) of the mixed phase, we take first the limit $\lambda_0 \to \infty$, and then the limit $\gamma \to \infty$, which results in the characteristic Josephson penetration length γd. In the anisotropic 'frozen' superconductor model for the Meissner phase [8], on the other hand, the order of the limits is reversed, hence the characteristic length scale λ_0. Both models, however, obtain the same expression for the parallel lower critical field (33) near criticality up to a numerical constant. In particular, we predict that $H_{c1}^\parallel(T)$ vanishes exponentially as temperature T approaches the decoupling transition from below, which implies the existence of an inflection point below T_* in this temperature profile (see figure 2).

In the presence of external parallel magnetic field, we expect that (double) layered extreme type-II superconductors follow a crossover phenomenon as a function of this field in the intermediate regime of the mixed phase. In particular, equations (39) and (41) indicate (with Λ_0 replaced by λ_J) that the parallel magnetization has the form $-4\pi M_\parallel = H_{c1}^\parallel(T) f[B_\parallel / B_*^\parallel(T)]$, where the latter functional dependence with parallel magnetic induction is plotted in figure 1. Note that in the mixed phase at low magnetic induction, $B_\parallel \ll B_*^\parallel$, we have that $-4\pi M_\parallel \lesssim H_{c1}(T)$. Hence, the parallel magnetization inherits the inflection of the parallel lower-critical field as a function of temperature $T \lesssim T_*$ at fixed $B_\parallel$. Figure 2 shows the phase diagram expected of a (double) layered superconductor in parallel external magnetic field near the critical regime discussed above. Formula (50) for the crossover field has been employed here, where the Josephson penetration length $\lambda_J(T)$ is interpolated between its behaviour at criticality (31) and its low-temperature value of γd. Note that $B_*^\parallel$ is expected to be practically constant at low temperatures since the mass anisotropy parameter γ has no temperature dependence in this regime. We therefore expect the crossover field to exhibit an inflection point in its temperature profile, much like the parallel lower-critical field does. By equation (51), however, the ratio of $B_*^\parallel$ to $H_{c1}^\parallel$ should be larger at criticality with respect to zero temperature by a factor of $\lambda_L^2(T_*)/\lambda_L^2(0)$, which in the Ginzburg–Landau theory approximation is given by $4\pi T_0/\gamma T_* \sim T_{c0}/\delta T_*$. Lastly, in spite of the above crossover phenomenon, the parallel vortex lattice (and flux quantization) will persist up to the parallel upper-critical field. In the Landau–Ginzburg approximation, this field is set by the in-plane coherence length $\xi = \xi_0(1 - T/T_{c0})^{-1/2}$ and by the mass anisotropy to be $H_{c2}^\parallel = \gamma \Phi_0 / 2\pi\xi^2$, hence the inequality $B_*^\parallel \ll H_{c2}^\parallel$. The critical behaviour of $H_{c2}^\parallel(T)$ near the decoupling transition at T_*, however, remains unknown.

We shall now examine the various physical scales that arise from the present theory in the context of high-temperature superconductivity. The oxide superconductor $Bi_2Sr_2CaCu_2O_8$ may be classified as a layered superconductor with an extreme mass anisotropy [1], $\gamma \sim 100$. Assuming typical parameters $T_{c0} \sim 100$ K, $d = 15$ Å and $\lambda_0 \sim 10^3$ Å, we obtain from equation (58) that $\delta T_* \sim 0.5$ K. It is interesting to remark that the zero-temperature Josephson penetration length γd and the zero-temperature London penetration length, λ_0, are of the same order of magnitude in this material. The fact that the present estimate for the size of the critical region is smaller by an order magnitude with respect to the estimate based on the above-mentioned anisotropic 'frozen' superconductor model [8] is simply then a result of the numerical factor in equation (58). Both theories, on the other hand, predict an inflection point in the temperature profile $H_{c1}^{\parallel}(T)$ (see figure 2). Wan *et al* [12] also observe an inflection point in the field of first penetration H_p versus T for $Bi_2Sr_2CaCu_2O_8$, but in the perpendicular orientation. This may be a vestige of the same prediction made here for the parallel lower-critical field if geometrical demagnetization effects are presumed to be strong. In particular, consider the regime in temperature $T_c < T < T_*$, where the planes are resistive while remaining Josephson coupled, and hence where only parallel Josephson vortices exist [8]. Then the field of first penetration in the perpendicular orientation is limited only by those portions of the field lines that run *parallel* to the top and bottom layers of the sample. Clearly, direct measurements of the parallel lower critical field in the critical regime of these materials would be highly desirable. Finally, we mention that the parallel crossover field (50) at zero temperature, $B_*^{\parallel}(0) = (2/\pi^2)(\Phi_0/d^2\gamma)$, should be approximately 2 T for $Bi_2Sr_2CaCu_2O_8$, while it should be orders of magnitude smaller at temperatures just below the decoupling transition temperature. We therefore suggest that the parallel reversible magnetization be measured in a clean thin film of this material near criticality [27], where the crossover field is expected to be quite modest in magnitude. Note that the present theory is valid only for perpendicular components of magnetic induction with magnitude greater than the perpendicular crossover field [16] $B_*^{\perp} \sim \Phi_0/d^2\gamma^2$, which in the case of $Bi_2Sr_2CaCu_2O_8$ is approximately 1 kG.

5. Discussion

In summary, we find no evidence for field dependence in the decoupling transition temperature, T_*, of clean double-layered extreme type-II superconductors in the intermediate regime of the mixed phase. Since the double layer represents the weakest link, we believe that this result remains true in the general case of any finite number of layers as long as the interlayer coupling is weak enough so that the system remains effectively 2D; for example, for perpendicular fields above the Glazman–Koshelev 2D–3D crossover scale, $B_{\perp} \sim \Phi_0/\gamma^2 d^2$, which guarantees the absence of vortex loops that traverse many layers [16]. In general, the extreme type-II limit $Nd \ll \lambda_L$ must be taken first, however, so that magnetic screening effects may be neglected. Also, the effect of pinning centres is not expected to be relevant in the critical regime, since the Josephson penetration length $\lambda_J(T)$ diverges exponentially as temperature T approaches T_* from below. We do, however, find a crossover field above which the parallel reversible magnetization decays with parallel magnetic field $H_{\parallel}$ as $H_{\parallel}^{-3}$ in the double-layer case (see figures 1 and 2, and [27]). The array of Josephson vortices is nevertheless expected to persist up to the parallel upper-critical field. It is important to mention that the present double-layer study cannot account for effects due to the first-order commensuration transitions predicted to occur in the parallel vortex lattice with many layers [26].

In order to account for the entropy due to wandering of the Josephson vortices in the calculations reported above, we have considered the length dimension of the vortex as imaginary time, and proceeded to compute the corresponding 'quantum mechanical' correction to the 'mass'. Employing a renormalization-group improved semiclassical approximation to this end, we have found that the parallel lower-critical field vanishes exponentially as it approaches the decoupling transition temperature from below. Very similar results have been obtained recently by the author using an alternative anisotropic 'frozen' superconductor model that operates from the Meissner phase [8]. Note that although the dimensionless Planck constant (17) has a value of $\bar{\hbar}_F = 8\pi$ at the decoupling transition, which is far from being small, it is suspected that the above cited renormalization-group, improved semiclassical results are in fact exact for the case of a single Josephson vortex [24, 25]. Less is known, however, with respect to the validity of the semiclassical approximation in the critical regime for the case of the array. For example, we have computed the entropic pressure (49) of the array to first order in powers of the effective dimensionless Planck constant (17) and found it to be negligibly small at fields below a relatively large bare scale (43). However, we were unable to find a renormalization group for this contribution. Also, it has been argued in the literature that the pressure exerted between interfaces in two dimensions generally varies quadratically with temperature [39], which translates into a second-order correction in the present semiclassical approximation. These effects generally stiffen the array of Josephson vortices and add a diamagnetic contribution to the high-field tail shown by the parallel magnetization (see figure 1 and [27]). However, any such additional diamagnetic correction does not affect the conclusion drawn here that no layer decoupling transition occurs as a function of external magnetic field in extreme type-II layered superconductors since it only results in a *stiffer* vortex lattice.

Acknowledgments

The author is grateful to D Dominguez, L Bulaevskii, B Ivlev, M Maley, and H Safar for very useful discussions. This work was performed under the auspices of the US Department of Energy, and was supported in part by National Science Foundation grant DMR-9322427.

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